Exercise 4

Use power series to solve the differential equation.

$$(x-3)y' + 2y = 0$$

Solution

$x = 0$ is an ordinary point, so the ODE has a power series solution centered here.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate the series with respect to x.

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Substitute these formulas into the ODE.

$$(x-3)\sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

Expand the first term.

$$x\sum_{n=1}^{\infty} n a_n x^{n-1} - 3\sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

Bring each coefficient inside the respective summand.

$$\sum_{n=1}^{\infty} n a_n x^n - \sum_{n=1}^{\infty} 3n a_n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

Because of n in the summand, the first series can be started from $n = 0$.

$$\sum_{n=0}^{\infty} n a_n x^n - \sum_{n=1}^{\infty} 3n a_n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

Make the substitution $n = k$ in the first series, the substitution $n = k + 1$ in the second series, and the substitution $n = k$ in the third series.

$$\sum_{k=0}^{\infty} k a_k x^k - \sum_{k+1=1}^{\infty} 3(k+1)a_{k+1}x^{(k+1)-1} + \sum_{k=0}^{\infty} 2a_k x^k = 0$$

Simplify the second series.

$$\sum_{k=0}^{\infty} k a_k x^k - \sum_{k=0}^{\infty} 3(k+1)a_{k+1}x^k + \sum_{k=0}^{\infty} 2a_k x^k = 0$$

Now that all the sums start from $k = 0$ and have x^k in the summand, they can be combined.

$$\sum_{k=0}^{\infty} \left[k a_k - 3(k+1)a_{k+1} + 2a_k\right] x^k = 0$$

The only way this infinite series can be zero is if the quantity in square brackets is zero.

$$ka_k - 3(k+1)a_{k+1} + 2a_k = 0$$

Solve for a_{k+1}.

$$(k+2)a_k - 3(k+1)a_{k+1} = 0$$

$$a_{k+1} = \frac{k+2}{3(k+1)}a_k$$

In order to determine a_k, plug in values for k and try to find a pattern.

$$k=0: \quad a_1 = \frac{0+2}{3(0+1)}a_0 = \frac{2}{3(1)}a_0$$

$$k=1: \quad a_2 = \frac{1+2}{3(1+1)}a_1 = \frac{3}{3(2)}\left[\frac{2}{3(1)}a_0\right] = \frac{3\cdot 2}{6\cdot 3}a_0$$

$$k=2: \quad a_3 = \frac{2+2}{3(2+1)}a_2 = \frac{4}{3(3)}\left(\frac{3\cdot 2}{6\cdot 3}a_0\right) = \frac{4\cdot 3\cdot 2}{9\cdot 6\cdot 3}a_0$$

$$\vdots$$

The general formula is

$$a_m = \frac{(m+1)!}{(3m)!!!}a_0 = \frac{(m+1)!}{3^m m!}a_0 = \frac{(m+1)m!}{3^m m!}a_0 = \frac{m+1}{3^m}a_0.$$

Therefore, the general solution is

$$\begin{aligned} y(x) &= \sum_{m=0}^{\infty} a_m x^m \\ &= \sum_{m=0}^{\infty} \frac{m+1}{3^m}a_0 x^m \\ &= a_0 \sum_{m=0}^{\infty} \frac{m+1}{3^m}x^m, \end{aligned}$$

where a_0 is an arbitrary constant.